

On a dynamical approach to an inverse problem in potential theory

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One of the classical problems in potential theory is to specify a closed surface Γ for a prescribed electric charge density μ in such a way that the uniform distribution of electric charges on Γ produces the same potential as μ (at least in a neighborhood of the infinity). Mathematically, this problem can be formulated as follows: given a measure μ , find a surface $\Gamma = \partial\Omega$ such that

$$(1) \quad \int h d\mu = \int_{\partial\Omega} h d\mathcal{H}^{N-1}$$

holds for all harmonic functions h defined in a neighborhood of $\bar{\Omega}$, where $\mathcal{H}^{N-1}$ denotes the $(N-1)$ -dimensional Hausdorff measure. We call such a surface $\partial\Omega$ a quadrature surface of μ . The mean value property of harmonic functions implies that (1) is valid when $\mu = \omega_N \delta_0$ and $\Omega = B(0, 1)$, where ω_N is the area of the unit sphere $\partial B(0, 1)$ in $\mathbb{R}^N$ and δ_0 is the Dirac measure supported at the origin. Thus, the identity (1) can be seen as a generalization of the mean value formula for harmonic functions.

Another equivalent formulation of the problem is the following overdetermined problem:

$$(2) \quad \begin{cases} -\Delta u = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial n} = -1 & \text{on } \partial\Omega, \end{cases}$$

where n is the outward unit normal vector to $\partial\Omega$. Namely, the boundary value problem (2) possesses a solution u in Ω if and only if the boundary $\partial\Omega$ satisfies the identity (1). From this point of view, the uniqueness of a quadrature surface $\partial\Omega$ holds in the case where $\mu = \omega_N \delta_0$ by a symmetry argument known as the method of moving planes or a simpler argument [6] based on careful observation of the profile of a radially symmetric solution $u = u_{\text{rad}}$ (the latter can also be applied to a heterogeneous situation and the main result below also holds for a weighted mean value formula).

The existence of a quadrature surface Γ of a prescribed μ has been investigated by several authors with different approaches (see [2], [1], [4] and [3]). In particular, a variational method was successfully applied to obtain a general existence result. However, as a counterexample by Henrot [4] shows, the uniqueness of a quadrature surface cannot hold for general μ , unlike the case $\mu = \omega_N \delta_0$.

Our main purpose here is to establish the uniqueness of Ω for some restricted class of measures μ . Namely, we prove that, if μ is sufficiently close to the Dirac measure $\omega_N \delta_0$, then there exists a unique smooth domain admitting (1). Moreover, we derive a quantitative estimate for the deviation of the domain from a ball, which exhibits the stronger rigidity of the domain of the mean value formula for perturbations of the Dirac measure of higher-order symmetry.

To state our main result, let us define the class $\mathfrak{M}_k$ of positive measures by

$$\mathfrak{M}_k := \left\{ \nu \mid \|\nu\|_{\mathfrak{M}} = \omega, \int h d\nu = 0 \text{ for all } h \in \bigcup_{j=1}^k H_j \right\},$$

where $\|\nu\|_{\mathfrak{M}} := \int d\nu$ and H_j denotes the vector space of all real-valued homogeneous harmonic polynomials of degree $j \in \mathbb{N}$ on $\mathbb{R}^N$. The total variation norm of a signed measure ν is defined by

$$\|\nu\|_{\mathfrak{M}} := \|\nu_+\|_{\mathfrak{M}} + \|\nu_-\|_{\mathfrak{M}},$$

where $\nu = \nu_+ - \nu_-$ is the Jordan decomposition. We also introduce a quantitative way of describing the “distance” between a surface $\partial\Omega$ and $\mathbb{S}^{N-1}$. Let $C^d(\mathbb{S}^{N-1})$ denote the space of all d -times continuously differentiable functions ρ on $\mathbb{S}^{N-1}$ and let Ω_ρ be the star-shaped domain defined by

$$\Omega_\rho := \left\{ \left(1 + \rho \left(\frac{x}{|x|} \right) \right) x \mid x \in \mathbb{B} \setminus \{0\} \right\} \cup \{0\}$$

for $\rho \in C^d(\mathbb{S}^{N-1})$ satisfying $\rho(\zeta) > -1$. Then, the norm $\|\rho\|_{C^d(\mathbb{S}^{N-1})}$ represents how much $\partial\Omega_\rho$ deviates from $\mathbb{S}^{N-1}$ in the radial direction up to d -th derivatives.

Theorem 1. *There is $\eta_0 > 0$ such that, for any μ with $\|\mu - \omega\delta_0\|_{\mathfrak{M}} + (\text{diam supp } \mu)^{N-1} < \eta_0$,*

(A) *there exists a unique smooth domain Ω satisfying $\text{supp } \mu \subset \Omega$ and (1); and*

(B) *if $\mu \in \mathfrak{M}_k$ for $k \in \mathbb{N} \cup \{0\}$, then for any $\varepsilon > 0$ and $d \in \mathbb{N}$*

$$(3) \quad \|\rho\|_{C^d(\mathbb{S}^{N-1})} \leq C \left(\|\mu - \omega\delta_0\|_{\mathfrak{M}} + (\text{diam supp } \mu)^{N-1} \right)^{1 + \frac{k+1}{N-1} - \varepsilon}$$

holds, where $\Omega = \Omega_\rho$ and $C = C(\varepsilon, k, d) > 0$.

Remark 2. The uniqueness holds among all domains Ω satisfying the regularity assumption that $\partial\Omega$ is of class C^1 and satisfies the interior sphere condition, i.e., for each point $x \in \partial\Omega$ there is a ball $B \subset \Omega$ such that $x \in \partial B$.

Let us briefly explain the main idea of the proof. The first step is to construct a “nice” domain Ω_* satisfying (1) and (3). For this purpose, we derive a deformation flow which describes how the domain Ω deforms when the measure μ varies in a smooth manner. The solvability of the flow, which was proved by the author [5] through the investigation of the spectral properties of the linearized operator, enables us to construct Ω_* having the desired properties. Moreover, the dynamical structure of the flow will be clarified by explicitly characterizing invariant manifolds of the flow through infinitely many conserved quantities called harmonic moments, and thus a precise quantitative estimate (3) for the deformation of Ω when μ approaches the Dirac measure $\omega\delta_0$ will be obtained. We then proceed to proving that Ω satisfying (1) must be identical with Ω_* by a contradiction argument, which is based on the maximum principle applied to the overdetermined problem (2). Indeed, if $\Omega \neq \Omega_*$, then the global-in-time solvability of the deformation flow starting from the nearly circular domain Ω_* deduces the existence of a super or subsolution to (2), and this leads to a contradiction.

References

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