Analytic approaches to the Möbius energy: History and recent topics

By

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We consider the Möbius energy

$$\mathcal{E}(\boldsymbol{f}) = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \left(\frac{1}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^2} - \frac{1}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^2} \right) ds_1 ds_2$$

defined for a closed curve $\boldsymbol{f} : \mathbb{R}/\mathcal{L}\mathbb{R} \to \mathbb{R}^n$. Here $\mathcal{L}$ is total length of the closed curve, s_i's are arc-length parameters, and $\mathscr{D}$ is the distance along the curve. This energy was originally proposed by O'Hara in 1991 for $n = 3$ as one of energies of knots. Indeed he introduced the energies

$$\mathcal{E}_{\alpha,p}(\boldsymbol{f}) = \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \left(\frac{1}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|_{\mathbb{R}^n}^\alpha} - \frac{1}{\mathscr{D}(\boldsymbol{f}(s_1), \boldsymbol{f}(s_2))^\alpha} \right)^p ds_1 ds_2,$$

which are called *O'Hara's* energy. The density contains the negative power of "distance", which implies that a minimizer, if exists, is the "canonical configuration" of knots among the given knot type, even though it makes the analysis hard.

It is easy to see that $\mathcal{E}_{\alpha,p}$ is scale-invariant if $\alpha p = 2$, including our energy $\mathcal{E} = \mathcal{E}_{2,1}$. In mid-1990's, Freedman-He-Wang showed that $\mathcal{E}$ has the invariance not only under scaling but also under Möbius transformations. Since then, it has been called the *Möbius* energy.

In this talk I firstly survey fundamental results on the Möbius energy:

- existence of minimizers in the class of prime knots,
- Kusner-Sullivan conjecture,
- bi-Lipschitz continuity,
- regularity of critical points,
- gradient flow

etc.

Secondary I will give recent progress of analytic approaches to $\mathcal{E}$. Blatt [1] found the proper domain of the energy: $\mathcal{E}(\boldsymbol{f}) < \infty$ if and only if $\boldsymbol{f}$ is bi-Lipschitz and belongs to the fractional Sobolev space $H^{3/2} \cap H^{1,\infty}$. Consequently we may assume the existence of the unit tangent vector $\boldsymbol{\tau}(s) = \boldsymbol{f}'(s)$ almost everywhere. By use of the unit tangent vector field along the curve, the energy may be decomposed into three parts:

$$\mathcal{E}(\boldsymbol{f}) = \mathcal{E}_1(\boldsymbol{f}) + \mathcal{E}_2(\boldsymbol{f}) + 4,$$

where

$$\begin{aligned}
\mathcal{E}_i(\boldsymbol{f}) &= \iint_{(\mathbb{R}/\mathcal{L}\mathbb{Z})^2} \mathscr{M}_i(\boldsymbol{f})\, ds_1 ds_2, \\
\mathscr{M}_1(\boldsymbol{f}) &= \frac{\|\boldsymbol{\tau}(s_1) - \boldsymbol{\tau}(s_2)\|^2_{\mathbb{R}^n}}{2\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|^2_{\mathbb{R}^n}}, \\
\mathscr{M}_2(\boldsymbol{f}) &= \frac{2}{\|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|^4_{\mathbb{R}^n}} \\
&\qquad \times \det \begin{pmatrix} \boldsymbol{\tau}(s_1)\cdot\boldsymbol{\tau}(s_2) & (\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2))\cdot\boldsymbol{\tau}(s_1) \\ (\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2))\cdot\boldsymbol{\tau}(s_2) & \|\boldsymbol{f}(s_1) - \boldsymbol{f}(s_2)\|^2_{\mathbb{R}^n} \end{pmatrix}.
\end{aligned}$$

This was recently shown by our research group [2]. The first decomposed energy $\mathcal{E}_1$ is an analogue of the Gagliardo semi-norm of $\boldsymbol{\tau}$ in the fractional Sobolev space $H^{1/2}$. This implies the domain of $\mathcal{E}$ is $H^{3/2} \cap H^{1,\infty}$, as shown by Blatt. The integrand $\mathscr{M}_2$ of second one has the determinant structure, which implies a cancellation of integrand. In [2] the Möbius invariance of each $\mathcal{E}_i$ has been also proved.

Since the last part "4" is an absolute constant, we can ignore it when considering variational problem. This fact shortens the derivation of variational formulae, and enables us to find their "good" estimates in several functional spaces [3]. Furthermore we find the L^2-gradient of each decomposed energy which contains the fractional Laplacian $(-\Delta_s)^{\frac{3}{2}}$ as the principal term [4].

Finally I will propose an analytic approach to Kusner-Sullivan conjecture using gradient flow.

References

[1] S. Blatt, *Boundedness and regularizing effects of O'Hara's knot energies*, J. Knot Theory Ramifications **21** (2012), 1250010, 9 pp.

[2] A. Ishizeki & T. Nagasawa, *A decomposition theorem of the Möbius energy I: Decomposition and Möbius invariance*, Kodai Math. J. **37** (2014), 737–754.

[3] A. Ishizeki & T. Nagasawa, *A decomposition theorem of the Möbius energy II: Variational formulae and estimates*, submitted.

[4] A. Ishizeki & T. Nagasawa, *The L^2-gradient of decomposed Möbius energies*, in preparation.